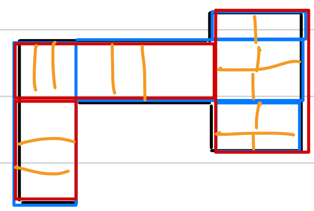
Math 564: Advance Analysis 1
Lecture 3

Last time we proved:

Claim (a). If a cylinder is partitioned into cylinders of fixed base-length, then $\tilde{\mu}_p$ on this partition is finitely additive.

Claim (b). For any two finite partitions into cylinders $\mathcal{P}$ and $\mathcal{Q}$ of a clopen set $U \in \mathcal{A}$, we have

$$\sum_{P \in \mathcal{P}} \tilde{\mu}_p(P) = \sum_{Q \in \mathcal{Q}} \tilde{\mu}_p(Q).$$



Proof. Let $\mathcal{R}$ be a common refinement of the partitions $\mathcal{P}$ and $\mathcal{Q}$, i.e. each $R \in \mathcal{R}$ is a cylinder contained in a piece in $\mathcal{P}$ and in a piece in $\mathcal{Q}$ s.t. $\bigcup \mathcal{R} = U$. Let $n \geq$ base-length(R) for all $R \in \mathcal{R}$, and refine $\mathcal{R}$ further to a partition $\mathcal{R}'$ into cylinders of base length n. Then

$$\sum_{P \in \mathcal{P}} \tilde{\mu}_p(P) \overset{(a)}{=} \sum_{P \in \mathcal{P}} \sum_{\substack{R \in \mathcal{R}' \\ R \subseteq P}} \tilde{\mu}_p(R) = \sum_{R \in \mathcal{R}'} \tilde{\mu}_p(R) = \sum_{Q \in \mathcal{Q}} \sum_{\substack{R \in \mathcal{R}' \\ R \subseteq Q}} \tilde{\mu}_p(R) \overset{(a)}{=} \sum_{Q \in \mathcal{Q}} \tilde{\mu}_p(Q).$$

$\square$

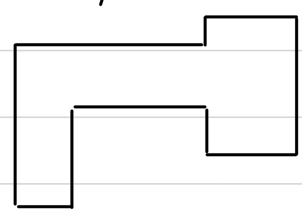
This shows that μ_p is well-defined on $\mathcal{A}$ and also that μ_p is finitely additive.

Claim (c). μ_p is automatically ctbly additive because no clopen set $A \in \mathcal{A}$ can be partitioned into infinitely many other nonempty clopen sets.

Proof. If $A = \bigsqcup_n \overset{\neq \emptyset}{A_n}$, A and the A_n clopen, then A is compact and $\{A_n\}_{n \in \mathbb{N}}$ is an open cover, hence has a finite subcover, contradicting that A_n are disjoint and nonempty. $\square$

This shows that μ_P is a premeasure on clopen sets.

Premeasure on box-algebra in $\mathbb{R}^d$. Let $\mathcal{A}$ be the algebra generated by boxes in $\mathbb{R}^d$, hence each $A \in \mathcal{A}$ is a finite disjoint union of boxes.



$A \in \mathcal{A}$

For an interval I,

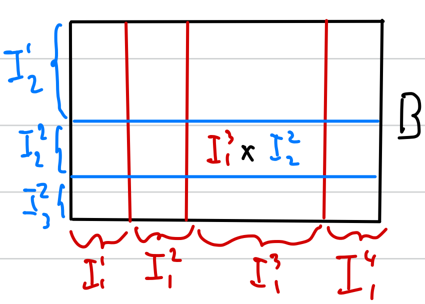
$$\mathrm{lh}(I) := \mathrm{right}(I) - \mathrm{left}(I),$$

which could be infinite. Define $\tilde{\lambda}$ on boxes by:

$$\tilde{\lambda}(I_1 \times I_2 \times \ldots \times I_d) := \mathrm{lh}(I_1) \cdot \mathrm{lh}(I_2) \cdot \ldots \cdot \mathrm{lh}(I_d).$$

Extend this to $\mathcal{A}$ by $\lambda(A) := \sum_{n<k} \tilde{\lambda}(B_n)$, where $A \in \mathcal{A}$ and $\{B_n\}_{n<k}$ is a partition of A into boxes. We need to show that μ is well-defined and we do so the same way, using grid-partitions instead of partitions into same base-length cylinders.

A grid-partition of a box $B = I_1 \times I_2 \times \ldots \times I_d$ is a finite partition P into boxes of the following form: partition each $I_k = \bigsqcup_{n<N_k} I_k^n$, and take



$$P := \{ I_1^{n_1} \times I_2^{n_2} \times \ldots \times I_d^{n_d} : (n_1, n_2, \ldots, n_d) \in N_1 \times N_2 \times \ldots \times N_d \},$$

where we treat $N = \{0, 1, \ldots, N-1\}$.

Claim (a). $\tilde{\lambda}$ is additive on grid-partitions of a box, i.e. for a box B and a grid-partion $\mathcal{P}$ of B,

$$\tilde{\lambda}(B) = \sum_{P \in \mathcal{P}} \hat{\lambda}(P).$$

Proof: Follows by induction on d, using the distributivity law $(a_1 + \dots + a_n) \cdot (b_1 + \dots + b_m) = \sum_{i \le n, j \le m} a_i b_j$. $\square$

Claim (b). Let $\mathcal{P}$ & $\mathcal{Q}$ be finite partitions of a set $A \in \mathcal{A}$. Then

$$\sum_{P \in \mathcal{P}} \tilde{\lambda}(P) = \sum_{Q \in \mathcal{Q}} \hat{\lambda}(Q).$$

Proof. The same as for Claim (b) for clopens in $2^{\mathbb{N}}$ — HW. $\square$

This shows that λ is well-defined and finitely additive on $\mathcal{A}$. To show ctbl additivity, let's first record general properties of finitely additive measures.

Prop. Let μ be a fin. additive measure on an algebra $\mathcal{A}$. Then it is:

(i) monotone: $A \subseteq B \Rightarrow \mu(A) \le \mu(B)$ for all $A, B \in \mathcal{A}$.

(ii) ctbly supadditive: $\mu(\bigsqcup_{n \in \mathbb{N}} A_n) \ge \sum_{n \in \mathbb{N}} \mu(A_n)$ for all pairwise disjoint $A_n \in \mathcal{A}$ with $\bigsqcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

(iii) $\mu(\bigcup_{n < k} A_n) \le \sum_{n < k} \mu(A_n)$ for all $A_1, \dots, A_k \in \mathcal{A}$.

Proof. (i) $\mu(B) = \mu(A \sqcup (B \setminus A)) = \mu(A) + \mu(B \setminus A) \ge \mu(A)$.

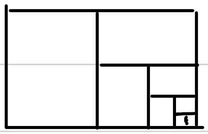
(ii) $\mu(\bigsqcup_{n \in \mathbb{N}} A_n) \ge \mu(\bigsqcup_{n < N} A_n) = \sum_{n < N} \mu(A_n) \nearrow_{N \to \infty} \sum_{n \in \mathbb{N}} \mu(A_n)$.

(iii) $\bigcup_{n < k} A_n = \bigsqcup_{n < k} A_n'$, where $A_n' := A_n \setminus (\bigcup_{i < n} A_i)$, we call this disjointification.

Then $\mu(\bigcup_{n < k} A_n) = \mu(\bigsqcup_{n < k} A_n') = \sum_{n < k} \mu(A_n') \underset{\text{by (a)}}{\le} \sum_{n < k} \mu(A_n)$. $\square$

Claim (c). λ is ctbly additive.

Proof. First let's assume B is a bounded box and $B = \bigsqcup_{n\in\mathbb{N}} B_n$, where the B_n are disjoint boxes. We wish that B was closed (hence compact) and the B_n were open (hence form an open cover).



We replace B by a closed box $B' \subseteq B$ s.t. $\lambda(B\setminus B') < \frac{\varepsilon}{2}$. We also replace each B_n by an open box $B_n' \supseteq B_n$ s.t. $\lambda(B_n'\setminus B_n) < \frac{\varepsilon}{2\cdot 2^{n+1}}$. Now $\lambda(B) \approx_{\varepsilon/2} \lambda(B')$ and $\sum_{n\in\mathbb{N}} \lambda(B_n') \approx_{\varepsilon/2} \sum_{n\in\mathbb{N}} \lambda(B_n)$.

Also B' is compact and $\{B_n'\}_{n\in\mathbb{N}}$ is an open cover of B' so it has a finite subcover $\{B_n'\}_{n<N}$. Then

$$\lambda(B) \approx_{\varepsilon/2} \lambda(B') \underset{\text{by (i)}}{\leq} \lambda\Big(\bigcup_{n<N} B_n'\Big) \underset{\text{by (iii)}}{\leq} \sum_{n<N} \lambda(B_n') \leq \sum_{n\in\mathbb{N}} \lambda(B_n') \approx_{\varepsilon/2} \sum_{n\in\mathbb{N}} \lambda(B_n).$$

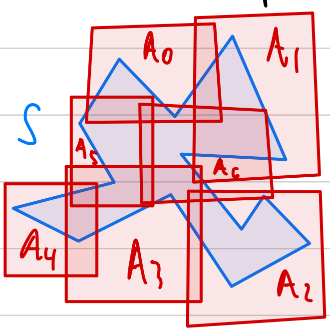
Thus, $\lambda(B) \leq \sum_{n\in\mathbb{N}} \lambda(B_n) + \varepsilon$, hence $\lambda(B) \leq \sum_{n\in\mathbb{N}} \lambda(B_n)$, so λ is ctbly sub-additive, and therefore ctbly additive by Property (ii) above.
The general case of when B is a finite union of potentially unbounded boxes reduces to the case we handled and is left as HW. $\square$

Having a premeasure on an algebra $\mathcal{A}$, we would like to get a measure on $\langle \mathcal{A} \rangle_\sigma$ and this can always be done:

Carathéodory's extension theorem. Every premeasure μ on an algebra $\mathcal{A}$ admits an extension to a measure on $\langle \mathcal{A} \rangle_\sigma$.
If μ is σ-finite, then this extension is unique and still denote it μ.

To prove this, we need the following notion:

Def. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a collection containing $\emptyset$ and covering X. Let $m : \mathcal{A} \to [0, \infty]$. The outer measure induced by m is the map $m^* : \mathcal{P}(X) \to [0, \infty]$ defined by



$$m^*(S) := \inf \left\{ \sum_{n \in \mathbb{N}} m(A_n) : \{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A},\ \bigcup_{n \in \mathbb{N}} A_n \supseteq S \right\}.$$

Prop. The outer is:

(a) monotone: $A \subseteq B \implies m^*(A) \leq m^*(B)$ for all $A, B \in \mathcal{P}(X)$.

(b) ctbly subadditive: $m^*\left(\bigcup_{n \in \mathbb{N}} B_n\right) \leq \sum_{n \in \mathbb{N}} m^*(B_n)$ for all $B_0, B_1, \ldots \in \mathcal{P}(X)$.

(a+b) $m^*\left(\bigcup_{n \in \mathbb{N}} B_n\right) \leq \sum_{n \in \mathbb{N}} m^*(B_n)$.

Proof. HW.